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ON THE DYNAMIC IMPOSSIBILITY OF GRIOLI REGULAR PRECESSION IN THE MOTION OF A RIGID BODY SUSPENDED BY A ROD*

G.V. GORR and G.A. KONONYKHIN

The dynamic impossibility of Grioli precession /1/ in the motion of a heavy rigid body suspended by a rod is proved. Along with the regular precession relative to the vertical of a Lagrange gyroscope, there is well known /1/ to be regular precession of an asymmetric heavy rigid body relative to an inclined axis. This is also true for the motion of a heavy gyrostat /2/ and the motion of a heavy rigid body in a fluid /3/.

1. Formulation of the problem. The equations of motion of a heavy rigid body, suspended by a weightless rod, whose ends are clamped by spherical joints at a fixed point and the point of suspension, will be written in the vector form /4/, by introducing instead of the central tensor of inertia the tensor of inertia relative to the point of suspension

$$\begin{gathered}\mathbf{A}\cdot\boldsymbol{\omega}^* + m\rho^2[(\mathbf{e}\cdot\boldsymbol{\omega}^*)\mathbf{e} - \boldsymbol{\omega}^* + (\boldsymbol{\omega}\cdot\mathbf{e})(\boldsymbol{\omega}\times\mathbf{e})] = \\ (\mathbf{A}\cdot\boldsymbol{\omega})\times\boldsymbol{\omega} + R(t)\rho(\mathbf{e}\times\mathbf{r}) \\ mr_0[\mathbf{r}^{**} + \boldsymbol{\omega}^*\times\mathbf{r} + 2\boldsymbol{\omega}\times\mathbf{r}^* + (\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega} - \mathbf{r}\omega^2] = \\ m\rho[\mathbf{e}\times\boldsymbol{\omega}^* - (\boldsymbol{\omega}\cdot\mathbf{e})\boldsymbol{\omega} + \omega^2\mathbf{e}] + mg\mathbf{v} - R(t)\mathbf{r} \\ \mathbf{v}^* = \mathbf{v}\times\boldsymbol{\omega};\ \rho = |\overline{OC}|\end{gathered} \tag{1.1}$$

Eqs.(1.1) have the first integrals

$$\begin{gathered}mr_0^2(\mathbf{r}^* + \boldsymbol{\omega}\times\mathbf{r})^2 + 2mr_0\rho(\mathbf{r}^* + \boldsymbol{\omega}\times\mathbf{r})\cdot(\boldsymbol{\omega}\times\mathbf{e}) + (\mathbf{A}\boldsymbol{\omega}\cdot\boldsymbol{\omega}) - \\ 2mg\rho(\mathbf{e}\cdot\mathbf{v}) - 2mgr_0(\mathbf{r}\cdot\mathbf{v}) = 2E \\ \mathbf{v}\cdot\{\mathbf{A}\boldsymbol{\omega} + mr_0\rho[\mathbf{e}\times(\mathbf{r}^* + \boldsymbol{\omega}\times\mathbf{r}) + \mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{e})] + \\ mr_0^2[\mathbf{r}\times(\mathbf{r}^* + \boldsymbol{\omega}\times\mathbf{r})]\} = k\end{gathered} \tag{1.2}$$

Here, $\boldsymbol{\omega}$ is the angular velocity vector of the body of mass m, $\mathbf{v}$ is the unit vector indicating the direction of the gravity force, $\mathbf{r}$ is the unit vector directed along the rod of length r_0 from the fixed point O_1 to the point of suspension O, $R(t)$ is the size of the reaction force, $\mathbf{e}$ is the unit vector from the point O to the centre of mass C of the body, $\mathbf{A}$ is the tensor of inertia of the body, constructed at the point O, and E and k are constants of the integrals; the asterisk denotes the relative derivative.

We introduce in the fixed space the unit vector $\boldsymbol{\gamma}$:

$$\boldsymbol{\gamma}^* = \boldsymbol{\gamma}\times\boldsymbol{\omega} \tag{1.3}$$

Let $\varkappa$ be the angle between $\mathbf{v}$ and $\boldsymbol{\gamma}$; then

$$\mathbf{v}\cdot\boldsymbol{\gamma} = c_0,\ \boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = 1,\ \mathbf{v}\cdot\mathbf{v} = 1,\ c_0 = \cos\varkappa \tag{1.4}$$

We assume that the body performs regular precession relative to the vector $\boldsymbol{\gamma}$ in such a way that the angle between the barycentric axis through the vector $\mathbf{e}$ and the vector $\boldsymbol{\gamma}$ is constant throughout the motion (obviously, $\rho \neq 0$)

$$\mathbf{e}\cdot\boldsymbol{\gamma} = a_0 \tag{1.5}$$

Differentiating (1.5) in the light of Eq.(1.3), we obtain in the case of regular precession

$$\boldsymbol{\omega} = n_0\mathbf{e} + m_0\boldsymbol{\gamma} \tag{1.6}$$

where n_0 and m_0 are constants. Notice that, in Grioli's solution for the classical problem of the motion of a rigid body about a fixed point, $n_0 = m_0$. To preserve generality we shall assume henceforth that $n_0 \neq m_0$. After substituting (1.6) into (1.3) we have

$$\boldsymbol{\gamma}^* = n_0(\boldsymbol{\gamma}\times\mathbf{e}) \tag{1.7}$$

We connect a moving coordinate system with the vector e: $\mathbf{e} = (0, 0, 1)$. Then we satisfy relations (1.5), (1.7) and the condition $\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = 1$ by writing the components of the vector $\boldsymbol{\gamma}$ in the form

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$$\gamma_1 = a_0' \sin\varphi,\ \gamma_2 = a_0' \cos\varphi,\ \gamma_3 = a_0 \qquad (1.8)$$
$$(a_0 = \mathbf{e}\cdot\boldsymbol{\gamma} = \cos\theta_0,\ a_0' = \sin\theta_0,\ \varphi = n_0 t + \varphi_0)$$

where t is time. The kinematic Eq.(1.1) and the final relations for $\mathbf{v}$ of (1.4) are satisfied by putting

$$\mathbf{v} = (c_0 + a_0 b_0')\boldsymbol{\gamma} - b_0'\mathbf{e}\sin\psi - b_0'(\boldsymbol{\gamma}\times\mathbf{e})\cos\psi \qquad (1.9)$$
$$(b_0' = b_0/a_0',\ b_0 = \sin\varkappa,\ \psi = m_0 t + \psi_0)$$

2. Integration of the equations of rotation.

On multiplying the first Eq.(1.1) scalarly by $\mathbf{e}$, we obtain an equation which, on substituting into it expressions (1.6) and (1.8), must given identity in the variable φ. This requirement leads to the conditions (A_{ij} are the components of the inertia tensor)

$$A_{11} = A_{22},\ A_{12} = 0,\ A_{13}a_0 = 0,\ A_{23}a_0 = 0 \qquad (2.1)$$

If we assume that $a_0 \neq 0$, we obtain $A_{13} = A_{23} = 0$, and by the first two equations of (2.1), the body will be a Lagrange gyroscope. Since the problem of finding the conditions for the existence of Grioli precessions has been posed, it follows from (2.1) that $a_0 = 0$, i.e., the angle between $\mathbf{e}$ and $\boldsymbol{\gamma}$ is $\pi/2$. Relations (1.8) and (1.9) then take the form

$$\gamma_1 = \sin\varphi,\ \gamma_2 = \cos\varphi,\ \gamma_3 = 0 \qquad (2.2)$$
$$\mathbf{v} = c_0\boldsymbol{\gamma} - b_0\mathbf{e}\sin\psi - b_0(\boldsymbol{\gamma}\times\mathbf{e})\cos\psi$$

By (1.6) and (2.2), the components of the angular velocity vector in the moving system are

$$\omega_1 = m_0\sin\varphi,\ \omega_2 = m_0\cos\varphi,\ \omega_3 = n_0 \qquad (2.3)$$

For the components of the tensor A_{ij} we introduce the following notation (we can assume without loss of generality that $A_{23} = 0$):

$$A_{11} = A_{22} = A,\ A_{13} = B,\ A_{33} = C \qquad (2.4)$$

Denote by r_1, r_2, r_3 the components of the unit vector $\mathbf{r}$ in the moving coordinate system. Then substitution of (2.2)-(2.4) into the first equation of (1.1) gives (σ is an auxiliary variable)

$$r_1 = \sin\sigma\,[B(n_0^2 - m_0^2\sin^2\varphi) - m_0 n_0 C\sin\varphi]/f(\varphi) \qquad (2.5)$$
$$r_2 = -\sin\sigma\,(Bm_0\sin\varphi + n_0 C)\,m_0\cos\varphi/f(\varphi)$$
$$r_3 = \cos\sigma,\ R(t) = f(\varphi)/\rho\sin\sigma$$
$$f(\varphi) = [B^2 m_0^2(m_0^2 - 2n_0^2)\sin^2\varphi + 2m_0 n_0(m_0^2 - n_0^2)\cdot BC\sin\varphi + n_0^2(B^2 n_0^2 + C^2 m_0^2)]^{1/2} \qquad (2.6)$$

3. Integration of the second Eq.(1.1).

The vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ are specified in the semimoving basis $\mathbf{e}, \boldsymbol{\gamma}, \boldsymbol{\gamma}\times\mathbf{e}$ by (1.6) and (1.9), and have in this sense an invariant form. On the basis of (2.2) and (2.5), the vector $\mathbf{r}$ is also conveniently written in the similar form

$$\mathbf{r} = \alpha_1\mathbf{e} + \alpha_2\boldsymbol{\gamma} + \alpha_3(\boldsymbol{\gamma}\times\mathbf{e}) \qquad (3.1)$$
$$\alpha_1 = \cos\sigma,\quad \alpha_2 = \Phi(\varphi)\sin\sigma/f(\varphi)$$
$$\alpha_3 = Bn_0^2\cos\varphi\sin\sigma/f(\varphi)$$
$$\Phi(\varphi) = B(n_0^2 - m_0^2)\sin\varphi - m_0 n_0 C$$

Since $a_0 = 0$ the vectors $\mathbf{e}, \boldsymbol{\gamma}, \mathbf{e}\times\boldsymbol{\gamma}$ are mutually orthogonal. We substitute into the second equation of (1.1), relations (1.6), (1.9), (3.1) and project the resulting equation onto the vectors $\mathbf{e}, \boldsymbol{\gamma}, \mathbf{e}\times\boldsymbol{\gamma}$. We obtain (the prime denotes the time derivative)

$$mr_0(\alpha_1^{\cdot\cdot} - 2m_0\alpha_3^{\cdot} - m_0^2\alpha_1) = m\rho m_0^2 - mgb_0\sin\psi - \alpha_1 f(\varphi)/\rho\sin\sigma \qquad (3.2)$$
$$mr_0\alpha_2^{\cdot\cdot} = mgc_0 - \alpha_2 f(\varphi)/\rho\sin\sigma$$
$$mr_0(\alpha_3^{\cdot\cdot} + 2m_0\alpha_1^{\cdot} - m_0^2\alpha_3) = -mgb_0\cos\psi - \alpha_3 f(\varphi)/\rho\sin\sigma$$

Substituting for α_2 from (3.1) into the right-hand side of the second equation of system (3.2), we find $\alpha_2^{\cdot\cdot}$ in terms of the known function of time. Integrating this expression twice and noting that the function α_2 is bounded (it is the projection of a unit vector onto the vector $\boldsymbol{\gamma}$), we obtain

$$mr_0\rho\alpha_2 = \lambda_0 + Bn_0^{-2}(n_0^2 - m_0^2)\sin\varphi \qquad 3.3)$$
$$mg\rho c_0 + m_0 n_0 C = 0$$

Equating the expressions for α_2 of (3.1) and (3.3), we find the time dependence of σ:

$$\sin\sigma = (\beta_0 + \alpha_0/\Phi(\varphi))\, f(\varphi) \tag{3.4}$$
$$\beta_0 = (mr_0\rho n_0^2)^{-1}, \quad \alpha_0 = (\lambda_0 n_0^2 + m_0 n_0 C)/m\rho r_0 n_0^2$$

Then, by (3.1), $\alpha_3 = Bn_0^2\cos\varphi\,(\beta_0 + \alpha_0/\Phi(\varphi))$. Using this dependence, we can integrate the last equation of (3.2) and obtain ($\varkappa_0$ is an arbitrary constant)

$$\alpha_1 = \cos\sigma = \frac{m_0 n_0 \alpha_0}{2(n_0^2 - m_0^2)}\ln|\Phi(\varphi)| + \tag{3.5}$$
$$\frac{Bn_0^3[B(n_0^2 - m_0^2) - Cm_0 n_0 \sin\varphi]\,\alpha_0}{2m_0[\Phi(\varphi)]^2} + \frac{1}{2}Bn_0 m_0 \beta_0 \sin\varphi - \frac{gb_0\sin\psi}{2m_0^2 r_0} + \varkappa_0$$

We first take the special case in (3.5) when $n_0^2 = m_0^2$, i.e., $m_0 = \varepsilon n_0$ ($\varepsilon = \pm 1$). From (3.4) we find that

$$\sin\sigma/f(\varphi) = \xi_0, \quad \xi_0 = (mr_0\rho - \lambda_0 - C)\,\beta_0 \tag{3.6}$$

We then have from (3.1):

$$\alpha_1 = \cos\sigma, \quad \alpha_2 = -m_0 n_0 C\xi_0, \quad \alpha_3 = Bn_0^2\xi_0\cos\varphi \tag{3.7}$$

Substituting α_2 and α_3 of (3.7) into the third equation of system (3.2), we obtain (l_0 is an arbitrary constant)

$$\cos\sigma = l_1\sin\varphi + l_0 \tag{3.8}$$
$$l_1 = {}^1/_2\varepsilon\beta_0\,(2Bn_0^4\xi_0\rho - mgb_0\rho - Bn_0^2)$$

By (3.6), $\sin\sigma = \xi_0 n_0^2\sqrt{C^2 + B^2\cos\varphi}$. We substitute this expression and (3.8) into the identity $\sin^2\sigma + \cos^2\sigma = 1$. The following conditions must then be satisfied:

$$l_1^2 - \xi_0^2 n_0^4 B^2 = 0, \quad l_0 l_1 = 0, \quad l_0^2 + \xi_0^2 n_0^4\,(C^2 + B^2) - 1 = 0 \tag{3.9}$$

We consider the case $l_1 = 0$, $\xi_0 = 0$, $l_0 = \pm 1$. It follows from (3.1) that $\mathbf{r} = \pm\mathbf{e}$, i.e., the body moves with the rod like a single rigid body. It can be shown that in this case the system is at rest.

Let $l_0 = 0$. We substitute (3.7), where $\cos\sigma$ takes the value (3.8), into Eq.(3.2), and require that the resulting equation be an identity in t. Then, $\rho = 0$, which is impossible, by the statement of the problem. Hence $m_0^2 \neq n_0^2$ and on the basis of (3.4) and (3.5) the equation $\sin^2\sigma + \cos^2\sigma = 1$ gives

$$f^2(\varphi)\,[\beta_0 + \alpha_0/\Phi(\varphi)]^2 + \alpha_1^2 - 1 = 0 \tag{3.10}$$

where the dependence $\alpha_1 = \alpha_1(\varphi)$ is given in (3.5). Here we put $\varphi = \pi k$, $k \in Z$ (Z is the set of integers); then, from (3.10), we obtain the expression $\sin(m_0\pi k/n_0) = d_0$, where d_0 is a constant. For all values of k, this expression can only hold if $m_0 = nn_0$, where $n \in Z$. By a suitable choice of the fixed coordinate system, we can arrange for $\psi = n\varphi$. On getting rid of the denominator and expanding $\ln|\Phi(\varphi)|$ in a Taylor series, we require that (3.10) be an identity in φ. Since we have an infinite series for the logarithmic function, we obtain $\alpha_0 = 0$. On again considering Eq.(3.10) with $\alpha_0 = 0$, we can conclude that $\psi = \pm\varphi$, i.e., we return to the case considered above.

The sum up, in the problem of the motion of a heavy rigid body suspended by a rod, Grioli precessions are dynamically impossible.

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